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Note

A Note on the Binomial Drop Polynomial of a Poset

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Suppose $(P, \prec)$ is a poset of size n and $\pi: P \to P$ is a permutation. We say that π has a drop at x if $\pi(x) \prec x$. Let $\delta_P(k)$ denote the number of π having k drops, $0 \leqslant k < n$, and define the drop polynomial $\Delta_P(\lambda)$ by

$$\Delta_P(\lambda) := \sum_k \delta_P(k) \binom{\lambda + k}{n}.$$

Further, define the incomparability graph $I(P)$ to have vertex set P and edges ij whenever i and j are incomparable in P, i.e., neither $i \prec j$ nor $j \prec i$ holds. In this note we give a short proof that $\Delta_P(\lambda)$ is equal to the chromatic polynomial of $I(P)$.

1. INTRODUCTION

Suppose $(P, \prec)$ is a poset, i.e., P is a set (of size n) partially ordered by a transitive irreflexive relation $\prec$. Denote by $\mathrm{Sym}(P)$ the set of all permutations $\pi: P \to P$. We say that π has a drop (at x) if $\pi(x) \prec x$. Let $\delta_P(k)$ denote the number of $\pi \in \mathrm{Sym}(p)$ which have k drops. Define $\Delta_P(x)$, the *binomial drop polynomial* of P, by

$$\Delta_P(x) := \sum_{k=0}^{n-1} \delta_P(k) \binom{x + k}{n}.$$

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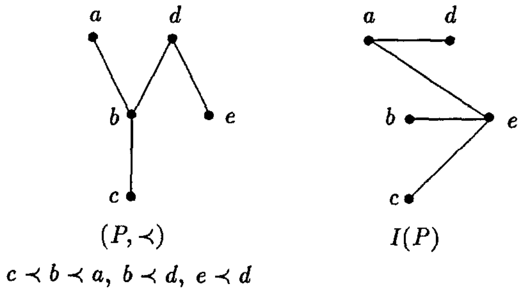

FIGURE 1

With $(P, \prec)$ one can also associate a graph $I(P)$, called the *incomparability graph* of P, as follows. The vertex set for $I(P)$ is just P; the edges of $I(P)$ are all pairs ij which are incomparable with respect to $\prec$, i.e., neither $i \prec j$ nor $j \prec i$ hold.

In Fig. 1, we show an example of a poset $(P, \prec)$ and $I(P)$.

Let G be a graph with vertex set V and edge set E, and let λ be a positive integer. By a proper λ-coloring of G we mean a map α: $V \to \{1, 2, ..., \lambda\}$ so that for each edge $uv \in E$, $\alpha(u) \neq \alpha(v)$. It is well known that the number of proper λ-colorings of G is given by a polynomial $\chi_G(\lambda)$ in λ, the so-called *chromatic polynomial* of G (e.g., see [4]). The main result in this note is the following.

THEOREM. *For any poset* $(P, \prec)$,

$$\Delta_P(\lambda) = \chi_{I(P)}(\lambda) \tag{1}$$

for all λ.

2. THE PROOF

Let us call a bijection β: $\{1, 2, ..., n\} \to P$ a *numbering* of $(P, \prec)$. We say that β has *descent* at i if $\beta(i+1) \prec \beta(i)$ in P. Clearly, a numbering can have at most $n-1$ descents and can never have a descent at n.

FACT. *There are exactly* $\delta_P(k)$ *numberings of* $(P, \prec)$ *which have* k *descents.*

Proof. We need to associate a unique numbering to each permutation. Arbitrarily extend $\prec$ to $\prec'$, a *linear* order on P. If $\pi \in \mathrm{Sym}(P)$ then define a numbering $\beta(\pi)$ by writing π as a product of disjoint cycles, with the largest (under $\prec'$) element first in each cycle, and then ordering the cycles so that their first elements are increasing (again, under $\prec'$). For example, if

$$\pi = (1\ 8\ 3)(2\ 4\ 7\ 5)(6\ 9)$$
$$= (7\ 5\ 2\ 4)(8\ 3\ 1)(9\ 6)$$
$$\text{then } \beta(\pi) = 7\ 5\ 2\ 4\ 8\ 3\ 1\ 9\ 6.$$

A standard argument (cf. [5]) shows that this map is a bijection and that the number of drops of π equals the number of descents of $\beta(\pi)$.

Now, if β is a numbering of $(P, \prec)$, let $C(\beta)$ denote the set of proper λ-colorings α: $P \to \{1, 2, ..., \lambda\}$ of $I(P)$, the incomparability graph of $(P, \prec)$, which satisfy

$$\alpha(\beta(i)) \leqslant \alpha(\beta(i+1)), \tag{2}$$

where *equality* is allowed in (1) only if β has a descent at i, i.e., $\beta(i+1) \prec \beta(i)$.

First, we claim that if α is a proper λ-coloring of $I(P)$ then there is always *some* numbering β such that $\alpha \in C(\beta)$. To see this, note that points having the *same* color are totally ordered (in $(P, \prec)$ by definition of a proper coloring of $I(P)$). To see this, simply arrange the elements of P into a sequence according to their colors with lowest color first, so that within a block of elements having the same color, the elements are arranged in increasing order in $(P, \prec)$.

Next, we claim that this numbering β is *unique*. Suppose not, i.e., suppose there are distinct numberings β and β' with $\alpha \in C(\beta) \cap C(\beta')$. Since $\beta \neq \beta'$, there must be points $p, q \in P$ such that p is *before* q in the numbering β, but such that p is *after* q in the numbering β'. In other words, there are integers i, j, k, l such that

$$\beta(i) = p, \qquad \beta(j) = q, \qquad i < j$$
$$\beta'(k) = q, \qquad \beta'(l) = p, \qquad k < l.$$

Since $\alpha \in C(\beta) \cap C(\beta')$ then

$$\alpha(p) = \alpha(\beta(i)) \leqslant \cdots \leqslant \alpha(\beta(j)) = \alpha(q)$$

and

$$\alpha(q) = \alpha(\beta'(k)) \leqslant \cdots \leqslant \alpha(\beta'(l)) = \alpha(p).$$

Hence, all of the above inequalities must in fact be *equalities*. However, by the definition of proper coloring this means that in $(P, \prec)$, we must have the *strict* inequalities

$$p \prec \cdots \prec q \qquad (\text{since } \alpha \in C(\beta))$$

and

$$q \prec \cdots \prec p \qquad (\text{since } \alpha \in C(\beta')),$$

which of course is impossible.

To summarize: if $\beta \neq \beta'$ then $C(\beta) \cap C(\beta') = \varnothing$, which in turn implies

$$\sum_{\beta} |C(\beta)| = \chi_{I(P)}(\lambda). \tag{3}$$

Finally, we claim that if β has k descents then

$$|C(\beta)| = \binom{\lambda + k}{n}. \tag{4}$$

For if a proper λ-coloring $\alpha \in C(\beta)$ actually uses only $j \leqslant \lambda$ colors then there are $n - j \leqslant k$ descents which can be identified in β as follows: insert the j colors in order, and allow replications (equality) at the identified descents. The number of ways to make these choices is $\binom{\lambda+k}{n}$ since we must choose a total of n objects from a set of $\lambda + k$ objects (the union of the λ colors and the positions of the k descents).

Combining the preceding facts now yields the desired conclusion:

$$\begin{aligned}\chi_{I(P)}(\lambda) &= \sum_{\beta} |C(\beta)| \\ &= \sum_{k=0}^{n-1} \sum_{\substack{\beta \text{ has} \\ k \text{ descents}}} |C(\beta)| \\ &= \sum_{k=0}^{n-1} \delta_P(k) \binom{\lambda + k}{n}.\end{aligned}$$

This proves the Theorem. ▮

Note that in the special case that $P = \{1, 2, ..., n\}$ is linearly ordered under the usual size order, $I(P)$ is the graph on n vertices with no edges, the $\delta_P(k)$ are just the Eulerian numbers (see [3]), and (1) reduces to the so-called Worpitzky identity:

$$\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{x + k}{n} = x^n. \tag{1'}$$

3. Concluding Remarks

1. The relationship between the numbers $\delta_P(k)$ and $\chi_{I(P)}(\lambda)$ can easily be inverted to give

$$\delta_P(n-k)=\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}\chi_{I(P)}(k-j). \tag{5}$$

In particular, this shows that δ_P only depends on $I(P)$, and not P. This is similar in spirit to the result that the number of linear extensions of a poset only depends on its *comparability* graph. (For a more general result, apparently due to P. Winkler, see the discussion on p. 194 of [5].) Are these results all special cases of some more general phenomenon? What other functions on a poset depend only on its (in-)comparability graph?

2. After hearing of our results from Richard Stanley, Einar Steingrímsson [6] reproved (1) using results from his thesis. Among many other things, he generalizes the notions of descents and drops (in his terminology, a mirror notion he calls "excedances") to certain wreath products of symmetric groups.

3. Typically, a result involving drops in a permutation has a corresponding companion result in which $\delta(k)$ is replaced by $\bar{\delta}(k)$, the number of permutations having k "weak" drops, i.e., occurrences of $\pi(x)\leqslant x$. In the case of (1), the companion result is

$$\bar{\Delta}_P(\lambda):=\sum_{k=0}^{n}\bar{\delta}_P(k)\binom{\lambda+k}{n}=\sum_{G\subseteq I(P)}\chi_G(\lambda),$$

where G ranges over *induced subgraphs* of $I(P)$.

This is a consequence of very general results on "Tutte-like" polynomials on digraphs which will appear in [2].

4. Our original motivation which led to (1) stemmed from encountering certain new classes of juggling pattern [1] where $(P,\prec)$ was $\{1, 2, ..., n\}$ (representing time) with the usual size order. Although we can give a natural interpretation of (1) as counting so-called "site swap" juggling patterns for certain "time" posets $(P,\prec)$, we are still not able to do this for an arbitrary poset.

Note added in proof. The authors have only recently become aware ofthe paper of J. Goldman, J. Joichi, and D. White, Rook theory III. Rook polynomials and the chromatic structure of graphs, *J. Combin. Theory Ser. B* **25** (1978), 135–142, which contains results that directly imply our results.

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